

CHARACTERIZING RIGHT-VEERING HOMEOMORPHISMS OF THE PUNCTURED TORUS VIA THE BURAU REPRESENTATION

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ABSTRACT. We classify right-veering homeomorphisms of the once-punctured torus using the Burau representation of the 3-strand braid group. We show that reducible and periodic mapping classes in B_3 can be identified as right-veering by consideration of the reduced version of the Burau representation. Given any element β in B_3 , we give a method to quickly determine its action on the generators of the fundamental group of the 3-times punctured disk. This action which determines whether β is right-veering, left-veering, or neither.

1. INTRODUCTION AND BACKGROUND

The goal of this paper is to use the Burau matrix of a 3-braid to determine whether the corresponding homeomorphism h is right-veering, left-veering, or neither. In Section 1, we give brief introductions to all relevant ideas used throughout this paper. In Section 2, we explain the veering information of a reducible or periodic element of B_3 by consideration of the trace of a power of its reduced Burau matrix. Finally, in Section 3, we prove the following main theorems that allow us to determine a 3-braid's action on the generators of the 3-punctured disk by consideration of their geometric intersection with arcs $\alpha_1, \alpha_2, \alpha_3$ that start and end at the punctures.

Theorem 1.1. *Given a 3-braid β and its matrix $M(\beta)$ in the image of the unreduced Burau representation, we can determine the geometric intersection number $i(\beta(y_i), \alpha_j)$ of $\beta(y_i)$ and α_j from the entries of $M(\beta)$ for each $1 \leq i, j \leq 3$.*

Theorem 1.2. *If β_1 and β_2 are 3-braids and $i(\beta_1(y_i), \alpha_j) = i(\beta_2(y_i), \alpha_j)$ for each i and j , then $\beta_1 = \Delta^{2k} \beta_2$ for some integer k .*

We start with a few definitions.

1.1. The Braid Group. The braid group B_n can be defined as the group generated by $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ with relations:

- (1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$, and
- (2) $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ if $|i - j| = 1$.

The braid group can also be described as a subgroup of the automorphism group of a free group or as a group of geometric objects up to isotopy. In this paper we will use the interpretation of B_n as the mapping class group of the space D_n , where D_n is the unit disk in \mathbb{R}^2 with the set $Q = \{p_1, p_2, \dots, p_n\}$ of n interior points of D^2 (assumed to lie regularly spaced on the x-axis) excised. We call these points *punctures*. We define the n -braid Δ_n to be

$$(\sigma_1 \sigma_2 \dots \sigma_{n-1})(\sigma_1 \sigma_2 \dots \sigma_{n-2}) \dots (\sigma_1 \sigma_2)(\sigma_1).$$

The square of Δ_n generates the infinite cyclic center of B_n . As a mapping class, Δ_n^2 is a full positive Dehn twist about a curve parallel to ∂D_n . We denote Δ_3 as Δ throughout this paper.

Let S be a genus one surface with one boundary component. We will make use of the fact that the group B_3 is isomorphic to the relative mapping class group of S , denoted here as $\text{Mod}(S, \partial S)$. This isomorphism is given by sending generators σ_1, σ_2 to positive Dehn twists T_x, T_y about dual closed curves x and y in S . We will also rely on the homomorphism ψ in the short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow B_3 \xrightarrow{\psi} SL_2(\mathbb{Z}) \longrightarrow 1.$$

By consideration of the representation onto $SL_2(\mathbb{Z})$, we use the Thurston classification of surface diffeomorphisms (which can be found in [8]) to determine whether a braid representative h is periodic, reducible, or pseudo-Anosov. We do this simply by calculating the trace. Say \hat{h} is the image of h under ψ . If $|\text{tr}(\hat{h})| = 2$, then h is reducible. If $|\text{tr}(\hat{h})| < 2$, h is periodic. Finally, if $|\text{tr}(\hat{h})| > 2$, h is pseudo-Anosov. We move forward using this classification to identify right-veering braids, however, first we must address the question of what a right-veering braid is.

1.2. Right-Veering Surface Homeomorphisms. Throughout, let Σ denote a compact, oriented surface with nonempty boundary. Let α and β be smooth properly embedded oriented arcs in Σ with a common basepoint $x \in \partial\Sigma$. We also assume that α and β intersect minimally and transversely. Consider the vectors v_α and v_β that are tangent to α and β , respectively, at x . We say that β is *to the right* of α if either α is isotopic to β or $\{v_\beta, v_\alpha\}$ corresponds to the orientation of Σ at x . Alternatively, if $\pi : \tilde{\Sigma} \rightarrow \Sigma$ is the universal cover of Σ . Let $\tilde{x} \in \partial\tilde{\Sigma}$ be a lift of x . Consider lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β with $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}$. The arc $\tilde{\alpha}$ divides $\tilde{\Sigma}$ into two regions. The region to the left where the boundary orientation induced from the region coincides with the orientation on $\tilde{\alpha}$ and the region to the right. Here, we say that β is *to the right* of α . We now define the monoid of right-veering maps.

Definition 1.3. *Let h represent an element of $\text{Mod}(\Sigma, \partial\Sigma)$. Then h is right-veering if for every $x \in \partial\Sigma$ and every properly embedded oriented arc α based at x , we have that $h(\alpha)$ is to the right of α . We denote the set of all such maps $\text{Veer}(\Sigma, \partial\Sigma) = \text{Veer}$.*

Similarly, we say that β is *to the left* of α if $\{v_\alpha, v_\beta\}$ corresponds to the orientation of Σ at x , and so we may also refer to a map h as *left-veering*. These homeomorphisms are especially helpful in contact topology. In [1] Honda, Kazez, Matić prove that a contact 3-manifold (M, ξ) is tight if and only if all of its compatible open book decompositions (Σ, h) have $h \in \text{Veer}$. Furthermore, they prove in [2] that if Σ has genus equal to one and one boundary component, (M, ξ) is tight if and only if some compatible open book has right-veering monodromy.

1.3. The Burau Representation. Now that we have an handle on what a right-veering map is, we introduce the tool that we will use to identify right-veering 3-braids: the Burau representation of B_n . The Burau representation can be defined in several different ways. The description given here comes from the viewpoint of covering spaces. Again, we let D_n be the n -punctured disc with basepoint p_0 on the boundary. Identify $\pi_1(D_n, p_0)$ in the usual way with the free group F_n , and let $\epsilon : \pi_1(D_n, p_0) \rightarrow \mathbb{Z}$ be the exponent sum map which takes the element $y_{r_1}^{\alpha_1} y_{r_2}^{\alpha_2} \dots y_{r_s}^{\alpha_s}$ to the integer $\sum_{i=1}^s \alpha_i$. Let (\tilde{D}_n, q) be the (regular) covering space of D_n which corresponds to the kernel of this homomorphism.

Concretely, \tilde{D}_n can be constructed by making a series of cuts from the boundary of D_n to each puncture (cut along the arcs δ_i described in the next section), then making a stack of \mathbb{Z} copies of this cut open space. At each component, glue the right side of each cut to the left side of the corresponding cut in the component directly above it. We end up with a space that resembles a parking garage with n “ramps” and infinitely many “levels”.

Since $\epsilon : \pi_1(D_n, p_0) \rightarrow \mathbb{Z}$ is a surjection, the group of deck transformations is isomorphic to $\mathbb{Z} = \langle \tau \rangle$. Letting $\Lambda = \mathbb{Z}[t, t^{-1}]$, the induced action τ_* of τ on $H_1(\tilde{D}_n)$ endows $H_1(\tilde{D}_n)$ with a Λ -module structure by setting $t \cdot \gamma = \tau_*(\gamma)$, for $\gamma \in H_1(\tilde{D}_n)$. Viewed as a Λ -module, $H_1(\tilde{D}_n)$ is free of rank $n - 1$. A chosen representative $h : D_n \rightarrow D_n$ will lift uniquely to a homeomorphism $\tilde{h} : \tilde{D}_n \rightarrow \tilde{D}_n$ that pointwise fixes the fiber over p_0 . Subsequently, \tilde{h} induces the Λ -module automorphism $\tilde{h}_* : H_1(\tilde{D}_n) \rightarrow H_1(\tilde{D}_n)$. We now make the following definition given a basis $\mathcal{B}_r = \{\gamma_1, \gamma_2, \dots, \gamma_{n-1}\}$ of $H_1(\tilde{D}_n)$.

Definition 1.4. *Given an element β of the braid group and an element h which represents it, the reduced Burau representation is the homomorphism:*

$$\rho_r : B_n \rightarrow GL_{n-1}(\Lambda)$$

where $\rho_r(\beta)$ equals the automorphism \tilde{h}_* . Let $M_r(\beta)$ denote the matrix for $\rho_r(\beta)$ with respect to the basis \mathcal{B}_r . We call this the reduced Burau matrix of β .

In a similar way, we may define the unreduced Burau representation. Let $H_1(\tilde{D}_n, \tilde{p}_0)$ be the first homology group of \tilde{D}_n relative to the subset $\tilde{p}_0 = q^{-1}(p_0)$. Here, \tilde{p}_0 denotes the full preimage of p_0 . We lift the set of generators y_1, y_2, \dots, y_n of $\pi_1(D_n, p_0)$ to a set of generators for $H_1(\tilde{D}_n, \tilde{p}_0)$. Now $H_1(\tilde{D}_n, \tilde{p}_0)$ can be thought of as a free Λ -module of rank n . With basis $\mathcal{B} = \{\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n\}$ of $H_1(\tilde{D}_n, \tilde{p}_0)$, we make the following definition.

Definition 1.5. *Given an element of the braid group β and a homeomorphism h that represents it, the unreduced Burau representation is the homomorphism*

$$\rho : B_n \rightarrow GL_n(\Lambda)$$

where $\rho(\beta)$ equals the automorphism $\tilde{h}_* : H_1(\tilde{D}_n, \tilde{p}_0) \rightarrow H_1(\tilde{D}_n, \tilde{p}_0)$. Let $M(\beta)$ denote the matrix for $\rho(\beta)$ with respect to the basis \mathcal{B} . We call this the unreduced Burau matrix of β .

It is a well-known fact that Burau is faithful for $n = 3$ (see [5]). In [4], Moody showed that Burau is not faithful for $n \geq 9$, and this was sharpened by Long and Paton in [3] to $n \geq 6$. The most recent improvement was made by Bigelow in [7] when it was shown that Burau is not faithful for $n = 5$.

1.4. A Geometric Interpretation. Unreduced Burau matrices can be obtained in several different ways, but perhaps the most beautiful approach was given by Moody in [4]. In this section we will describe his method for computing unreduced Burau matrices. Let y_i and δ_j for $i, j = 1, \dots, n$ be as shown in Figure 1. The line L divides the disk into upper and lower halves. We let ∂D_n^+ denote the upper half of ∂D_n . Define a map $\langle \cdot, \cdot \rangle : H_1(\tilde{D}_n, \partial D_n^+ \cup \tilde{Q}) \times H_1(\tilde{D}_n, \tilde{p}_0) \rightarrow \Lambda$ as follows. If $\tilde{\alpha}$ represents some homology class in $H_1(\tilde{D}_n, \tilde{p}_0)$, and $\tilde{\delta}_j$ is a lift of δ_j , we define

$$\langle \delta_j, \alpha \rangle = \sum_{k \in \mathbb{Z}} (t^k \tilde{\delta}_j, \tilde{\alpha}) t^k \in \Lambda$$

where $(t^k \tilde{\delta}_j, \tilde{\alpha})$ is the algebraic intersection number of the two arcs in \tilde{D}_n .

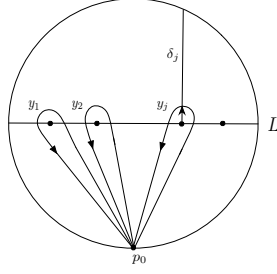
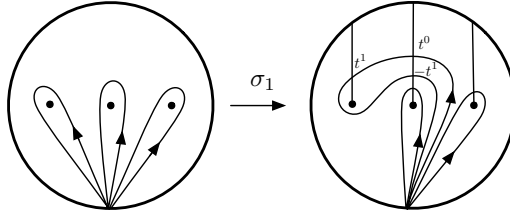


FIGURE 1. The n-punctured disk

It is clear that this pairing is only well-defined up to choice of the lifts and choice of orientation. We orient δ_j from the puncture to the boundary and orient y_i counterclockwise. Then orientations of the lifts are induced from the arcs in the base space. We use the following convention for choosing our lifts. Choose component \tilde{p}_0^* of \tilde{p}_0 , and declare the sheet of \tilde{D}_n containing \tilde{p}_0^* to be the 0-th level of \tilde{D}_n . Now, define \tilde{y}_j to be the unique lift of y_j that starts at \tilde{p}_0^* and ends at $t\tilde{p}_0^*$ and $\tilde{\delta}_j$ to be the unique lift of δ_j that intersects \tilde{y}_j . Now we can write the action of any braid β on basis vector \tilde{y}_j of $H_1(\tilde{D}_n, \tilde{p}_0)$ as:

$$\rho(\beta)\tilde{y}_j = \left(\langle \delta_1, \beta(y_j) \rangle\right)\tilde{y}_1 + \left(\langle \delta_2, \beta(y_j) \rangle\right)\tilde{y}_2 + \cdots + \left(\langle \delta_n, \beta(y_j) \rangle\right)\tilde{y}_n.$$

Using this method allows us to work in D_n in a very concrete way instead of \tilde{D}_n . The key is in keeping track of the level changes. For example, we see the action of the braid $\sigma_1 \in B_3$ on π_1 generators below in Figure 2. In particular, one can calculate the unreduced

FIGURE 2. The exponent of t indicates the level on which the intersection occurs.

matrix for $\sigma_1 \in B_3$ as

$$M(\sigma_1) = \begin{pmatrix} 0 & t & 0 \\ 1 & 1-t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, we can calculate $\sigma_2 \in B_3$ as

$$M(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & t \\ 0 & 1 & 1-t \end{pmatrix}.$$

2. REDUCIBLE AND PERIODIC CASES

In this section, we will use the reduced Burau representation to classify right-veering braids that are reducible or periodic. We do so largely by appealing to work done by Honda, Kazez, and Matić in [2] and by using the commutative diagram

$$\begin{array}{ccc} B_3 & \xrightarrow{\psi} & SL_2(\mathbb{Z}) \\ \rho_r \downarrow & \nearrow \pi & \\ \rho_r(B_3) & & \end{array}$$

In this diagram, π is specialization at $t = -1$, and ψ the quotient map from B_3 to B_3 modulo its infinite cyclic center as before. Geometrically, the homomorphism ψ can be seen as collapsing the boundary of S to a point. For a braid β , let $\widehat{\beta}$ denote its image under ψ in $SL_2(\mathbb{Z})$.

2.1. The Reducible Case. We use a characterization of reducible maps of S (see [2], for example) that says that any reducible map of S can be written as $T_a^k T_b^m$, where a is a boundary-parallel curve, and b is some nonseparating curve in S . Given the 2-fold branched cover of D_3 by S (the branch set being the three points shown below), we see that a half twist about an arc from p_i to p_{i+1} in D_3 lifts to a full Dehn twist about some nonseparating curve in S .

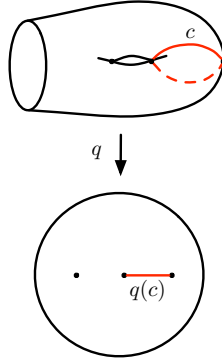


FIGURE 3. σ_2 lifts to T_c

It follows that reducible elements of B_3 differ from a conjugate of a power of σ_1 by a powers of Δ^2 . That is, any reducible braid ω can be written in the form $\Delta^{2k} \tau \sigma_1^m \tau^{-1}$, for some $\tau \in B_3$. We also use the following result of Honda, Kazez, and Matić that yields the veering information of a reducible map of S based upon the form above.

Lemma 2.1. (Honda, Kazez, Matić [2]) *Let $h = T_a^k T_b^m$ represent a reducible mapping class in $\text{Mod}(S, \partial S)$. Then h is right-veering if and only if either $k > 0$ or $k = 0$ and $m \geq 0$.*

Thus, given a reducible braid $\beta = \Delta^{2k} \tau \sigma_1^m \tau^{-1}$, our task is to extract the values k and m from the reduced Burau matrix $M_r(\beta)$, then use Lemma 2.1 to determine whether β is right-veering or not. We make several statements about the trace of $M_r(\beta)$. In particular,

we have the following proposition and corollaries which gives sufficient conditions for β to be right-veering.

Proposition 2.2. *Let $\beta \in B_3$ be reducible. Then $\text{tr}(\text{M}_r(\beta)) = t^{3k} + (-1)^m t^{3k+m}$.*

Proof. We have that the $\text{tr}(\text{M}_r(\sigma_1^m)) = 1 + (-1)^m t^m = \text{tr}(\text{M}_r(\tau \sigma_1^m \tau^{-1}))$. Now since $\text{M}_r(\Delta^{2k}) = \begin{pmatrix} t^{3k} & 0 \\ 0 & t^{3k} \end{pmatrix}$, we may conclude that

$$\text{tr}(\text{M}_r(\Delta^{2k} \tau \sigma_1^m \tau^{-1})) = t^{3k}(1 + (-1)^m t^m) = t^{3k} + (-1)^m t^{3k+m}.$$

□

Corollary 2.3. *Let $\beta \in B_3$ be reducible. If $\text{tr}(\text{M}_r(\beta)) = t^d - t^r$ for integers d and r , then $d = 3k$. If $\text{tr}(\text{M}_r(\beta)) = t^d + t^r$ is such that $3 \nmid r$, then $d = 3k$. In either of these cases, β is right-veering if and only if $d > 0$.*

Proof. Immediate from Proposition 2.2 and Lemma 2.1. □

Corollary 2.4. *Let $\beta \in B_3$ be reducible.*

- (1) *If $\text{tr}(\text{M}_r(\beta)) = t^d + t^r$ for nonnegative integers d and r , then β is right-veering.*
- (2) *If $\text{tr}(\text{M}_r(\beta)) = t^d + t^r$ for nonpositive integers d and r , then β is left-veering.*

Proof. If both d and r are nonnegative integers, then $k \geq 0$. If $k > 0$, then β is right-veering. If $k = 0$, then m is nonnegative which also yields β right-veering. An analogous argument is made if both d and r are nonpositive which gives β left-veering. □

Now the question becomes how does one tell if a reducible braid is right-veering if the trace is of the form $t^d + t^{-r}$ where d, r are positive integers both divisible by 3. For example, if $\text{tr}(\text{M}_r(\beta)) = t^6 + t^{-6}$, it is impossible to determine if $k = 2$ and $m = -12$ or if $k = -2$ and $m = 12$ by means already stated. Moreover, this distinction is necessary since the former implies β is right-veering and the latter that β is left-veering. The following theorem gives us sufficient information to determine k and m assuming $\text{tr}(\text{M}_r(\beta))$ and the eigenvalue λ for $\hat{\beta}$ are known.

Theorem 2.5. *Suppose that β is a nontrivial reducible map. Then value $|m|$ is the greatest common divisor of the entries off the diagonal of $\hat{\beta}$. Moreover, for $\lambda = 1$, the sign of m is the sign of the $(1, 2)$ -entry of $\hat{\beta}$, and for $\lambda = -1$, the sign of m is the sign of the $(2, 1)$ -entry of $\hat{\beta}$.*

Proof. We first suppose that $\hat{\beta}$ has eigenvalue $\lambda = 1$. For any $\tau \in B_3$, it is easily checked that $\hat{\beta}$ is of the form $\begin{pmatrix} 1 - amc & a^2m \\ -c^2m & 1 + acm \end{pmatrix}$, where $\hat{\tau} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $\hat{\tau}$ is in $SL_2(\mathbb{Z})$, we know that $\gcd(a, c) = 1$, and thus $\gcd(a^2, c^2) = 1$. Therefore, the greatest common divisor of a^2m and $-c^2m$ is $|m|$, and it is clear that the sign of m determines the sign of a^2m .

Now if $\lambda = -1$, $\hat{\beta}$ is of the form $\begin{pmatrix} amc - 1 & -a^2m \\ c^2m & -1 - acm \end{pmatrix}$. The same argument holds except now the sign of m determines the sign of c^2m . □

2.2. The Periodic Case. We use the fact that periodic maps of the torus have order dividing 12, and start by stating a key lemma.

Lemma 2.6. *For any positive integer n , h^n is right-veering if and only if h is right-veering.*

Proof. It is clear that if h is right-veering, then h^n is right-veering since Veer is a monoid. Now assume that $h^n(\alpha)$ is to the right of α for all properly defined arcs α . Then either $h(\alpha)$ is to the right of α for all relevant α , or there is some arc γ with $h(\gamma)$ to the left of γ . However, the latter case implies that $h^n(\gamma)$ is to the left of γ . This is a contradiction. \square

We have the following theorem which gives necessary and sufficient conditions for β^{12} to be right-veering.

Theorem 2.7. *Let $\beta \in B_3$ be a nontrivial periodic map. Then $M_r(\beta^{12}) = \begin{pmatrix} t^{6k} & 0 \\ 0 & t^{6k} \end{pmatrix}$ for some integer k . Moreover, β^{12} is right-veering if and only if $k \geq 0$.*

Proof. If β is a periodic braid, then $\hat{\beta}$ is a periodic element of $SL_2(\mathbb{Z})$. Since all periodic maps of T^2 have order dividing 12, β^{12} must be an element of $\ker(\psi)$ which is generated by Δ^4 . We know that Δ^{2k} are powers of Dehn twists about the boundary of D_3 , where this twist is positive if and only if k is positive. Since $M_r(\Delta^{2k}) = \begin{pmatrix} t^{3k} & 0 \\ 0 & t^{3k} \end{pmatrix}$, we have that $M_r(\beta^{12}) = \begin{pmatrix} t^{6k} & 0 \\ 0 & t^{6k} \end{pmatrix}$. Moreover, this element is a positive Dehn twist if and only if $k > 0$ and the identity element if and only if $k = 0$. \square

Corollary 2.8. *Let β be a periodic element of B_3 . Then β is right-veering if and only if $M_r(\beta^{12})$ has trace equal to $2t^s$ for some nonnegative integer s .*

Proof. The proof is immediate from the Lemma 2.6 and Theorem 2.7. \square

3. THE GENERAL CASE

In this section we use the unreduced matrix $M(\beta)$ of a 3-braid β to determine its veering information. In short, given $M(\beta)$, we use its entries to calculate geometric intersection numbers of $\beta(y_i)$ and arcs α_j for $1 \leq i, j \leq 3$ shown below in Figure 4. This allows us to determine the action of β on each y_i up to powers of Δ^2 . We can then write down a matrix $M(\beta')$ that represents this action using the method described in Section 1.4. The matrix $M(\Delta^{2k})M(\beta')$ will be $M(\beta)$, for exactly one integer k because of the faithfulness of the representation for $n = 3$. As long as $M(\beta')$ is well-chosen, the sign of k will tell us the veering information. We begin with the following theorem.

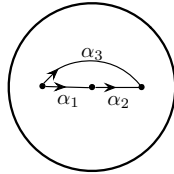


FIGURE 4.

Theorem 3.1. *Let $\beta \in B_n$. Then β is right-veering if and only if β sends every loop representing an element of $\pi_1(D_n, p_0)$ to the right.*

Proof. Assume β sends every loop representing an element of the fundamental group of D_n to the right. We endow D_n with a hyperbolic metric for which ∂D_n is geodesic. The universal cover $\pi : \tilde{X} \rightarrow D_n$ is a subset of the Poincaré disk D^2 . Consider the closure of \tilde{X} in D^2 , and denote one component of $\partial \tilde{X}$ as L . Without any loss of generality, we let γ be an oriented arc in D_n with endpoints $p_0 = \gamma(0)$ and $x_1 = \gamma(1)$ both in ∂D_n . We also assume that γ is essential in the sense that it does not cobound a digon with some subarc of ∂D_n . Then $\gamma \cup \alpha$ is a nontrivial loop based at p_0 , for α an embedded arc in ∂D_n oriented clockwise from x_1 to p_0 . Choose $\tilde{p}_0 \in \pi^{-1}(p_0)$ in L , and let $\tilde{\beta} : \tilde{X} \rightarrow \tilde{X}$ be a lift of β which fixes L pointwise.

Choose a lift $\tilde{\gamma} \cup \tilde{\alpha} = \tilde{\gamma} \cup \tilde{\alpha}$ of $\gamma \cup \alpha$ which has initial point \tilde{p}_0 and terminal point $\tilde{y}_0 = g(\tilde{p}_0)$, for $g \in \pi_1(D_n, p_0)$. Without loss of generality, we assume that $\tilde{\gamma}$ is geodesic. Note that $\tilde{\gamma}(1) = \tilde{x}_1 = \tilde{\alpha}(0)$. By hypothesis $\tilde{\beta}(\tilde{\gamma} \cup \tilde{\alpha})$ is in the region to the right of $\tilde{\gamma} \cup \tilde{\alpha}$. Because $\pi^{-1}(\alpha)$ is a collection of disjoint arcs in $\partial \tilde{X}$, we can conclude that $\tilde{\beta}(\tilde{\alpha}) \cap \tilde{\alpha} = \emptyset$. Therefore, $\tilde{\beta}(\tilde{\alpha})$ is in the region to the right of $\tilde{\alpha}$ and hence, $\tilde{\beta}(\tilde{\gamma})$ (having terminal point in $\tilde{\beta}(\tilde{\alpha})$) is in the region to the right of $\tilde{\gamma}$. The converse implication is clearly true by definition of Veer. \square

Given Theorem 3.1, our goal is to detect rightward movement of generators y_i of the fundamental group of D_n .

3.1. Proof of Theorem 1.1. Let β be a 3-braid and suppose that $C_i = (p_{1,i}(t), p_{2,i}(t), p_{3,i}(t))$ is the i -th column $M(\beta)$. Recall that $p_{j,i}(t) = \langle \delta_j, \beta(y_i) \rangle$. Define the following Laurent polynomials:

$$Q_i(t) = p_{1,i}(t) - tp_{2,i}(t), R_i(t) = p_{2,i}(t) - tp_{3,i}(t), S_i(t) = p_{1,i} - t^2 p_{3,i}(t).$$

We claim that Q_i, R_i, S_i are intersection pairings of $\beta(y_i)$ and $\alpha_1, \alpha_2, \alpha_3$, respectively. To see this, we first note that

$$Q_i(t) = \langle \delta_1, \beta(y_i) \rangle - t \langle \delta_2, \beta(y_i) \rangle = \langle \delta_1 - t^{-1} \delta_2, \beta(y_i) \rangle = \sum_{k \in \mathbb{Z}} (t^k (\tilde{\delta}_1 - t^{-1} \tilde{\delta}_2), \widetilde{\beta(y_i)}) t^k.$$

If $\tilde{\alpha}_1$ is a lift of α_1 to the 0-th level of \tilde{D}_n , then $\tilde{\delta}_1 - t^{-1} \tilde{\delta}_2$ and $\tilde{\alpha}_1$ represent the same element of $H_1(\tilde{D}_n, \partial \tilde{D}_n^+ \cup \tilde{Q})$. Scaling by t^{-1} is necessary because pushing $\tilde{\delta}_2$ off to the left results in a level change upwards. Similarly, $\tilde{\delta}_2 - t^{-1} \tilde{\delta}_3$ and $\tilde{\alpha}_2$ represent the same element of $H_1(\tilde{D}_n, \partial \tilde{D}_n^+ \cup \tilde{Q})$. Finally, if we say that $\tilde{\alpha}_3$ is the lift of α_3 that begins on the 0-th level of \tilde{D}_n (and, hence, ends on the -1 st level), then it is clear that $\tilde{\alpha}_3 = \tilde{\alpha}_1 + t^{-1} \tilde{\alpha}_2$ as homology classes. We have

$$\tilde{\alpha}_3 = (\tilde{\delta}_1 - t^{-1} \tilde{\delta}_2) + t^{-1} (\tilde{\delta}_2 - t^{-1} \tilde{\delta}_3) = \tilde{\delta}_1 - t^{-2} \tilde{\delta}_3 \in H_1(\tilde{D}_n, \partial \tilde{D}_n^+ \cup \tilde{Q}).$$

Thus, our formula for $S_i(t)$ gives us an intersection pairing of α_3 and $\beta(y_i)$.

It remains to show that there is no cancellation of terms in this pairing, and hence we are able to calculate the geometric intersection numbers of the aforementioned arcs by specializing each polynomial at $t = -1$. Let w_r be the r -th intersection of $\beta(y_i)$ and α_1 , and consider the loop that is the union of the arcs from w_r to w_{r+1} along α_1 and along $\beta(y_i)$. If $\beta(y_i)$ intersects α_1 in the same direction at both w_r and w_{r+1} , then this loop must contain exactly two of the punctures (this is a direct consequence of there being three punctures). Hence, if at w_r the lift $\widetilde{\beta(y_i)}$ is on level k of \tilde{D}_3 , then at w_{j+1} $\widetilde{\beta(y_i)}$ is on

level $k \pm 2$ of \widetilde{D}_3 . The arc $\beta(y_i)$ will intersect δ_1 or δ_2 either just before or just after each intersection with α_1 depending upon its orientation at w_r and w_{r+1} . Let's suppose $\beta(y_i)$ is oriented upward at w_r and w_{r+1} . Then there are three options for the diagram near w_r and w_{r+1} which are shown in Figure 5. The labeling in this figure reflects the monomial that is recorded in the row corresponding to either δ_1 or δ_2 . In all three instances, when

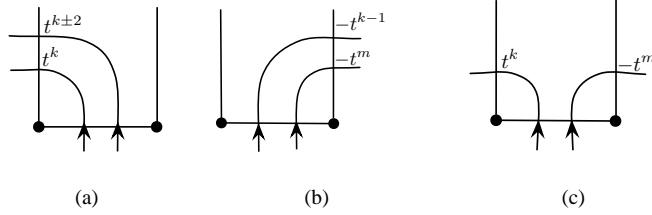


FIGURE 5. m is either $k - 3$ or $k + 1$.

we consider the monomial associated to δ_1 and $-t$ times the monomial associated to δ_2 , there is a consistency between the parity of the exponent of t and the sign of t (meaning same parity, same sign and opposite parity, opposite sign). As a result, specialization at $t = -1$ gives the accurate geometric intersection number of 2 in absolute value. A similar argument holds if $\beta(y_i)$ is oriented downward at w_r and w_{r+1} .

If $\beta(y_i)$ is oriented oppositely at w_r and w_{r+1} , then the loop described above must either contain exactly one puncture or possibly all three. Hence, if at w_r , the lift $\beta(y_i)$ is on level k , then at w_{r+1} it is either on level $k \pm 1$ or $k \pm 3$. The local picture looks like one of the three options shown in Figure 6. Again, we see consistency between parity of the exponent and the sign of t in the monomial associated to δ_1 and $-t$ times the monomial associated to δ_2 . So, indeed, $|Q_i(-1)|$ is the correct geometric intersection number. The same argument holds for $|R_i(-1)|$.

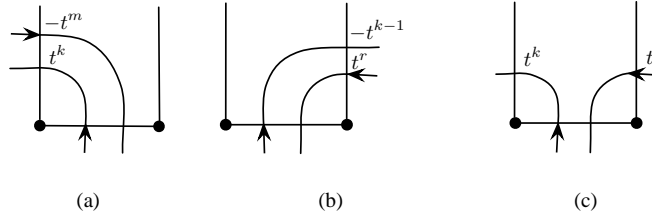


FIGURE 6. m is either $k \pm 1$ or $k \pm 3$ and r is either $k, k - 4$, or $k \pm 2$.

Lastly, the polynomials in the rows corresponding to δ_1 and δ_3 are inconsistent in the following sense: if the terms in the polynomial corresponding to δ_1 having even (odd) exponents are positive (negative), then the terms in the polynomial corresponding to δ_3 having even (odd) exponents are negative (positive). The only exceptions are the terms corresponding to intersections of the types seen below in Figure 7. However, multiplying the polynomial corresponding to δ_3 by $-t^2$ changes the any inconsistency to a consistency with the polynomial corresponding to δ_1 . Furthermore, summing with the polynomial corresponding to δ_1 cancels the terms not giving an intersection with α_3 as in Figure 7. Thus $|S_i(-1)|$ gives the geometric intersection number of $\beta(y_i)$ and α_3 .

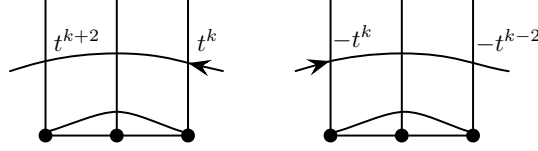


FIGURE 7.

Now, given these intersection numbers we argue that we can determine the action of β on each y_i up to full Dehn twists about the boundary.

3.2. Proof of Theorem 1.2. Let $\beta \in B_3$. Since β is uniquely determined by its action on the loops y_1, y_2, y_3 , we argue that the image of each y_i under β can be determined up to the action of the braid Δ^{2k} and isotopy given the geometric intersection numbers $i(\beta(y_i), \alpha_1) = a_i$, $i(\beta(y_i), \alpha_2) = b_i$, and $i(\beta(y_i), \alpha_3) = c_i$. Without any loss of generality, we consider the image $\beta(y_1)$. The arc $\beta(y_1)$ is the union of subarcs inside and outside the triangle formed by $\alpha_1, \alpha_2, \alpha_3$. Two arcs of particular interest are the initial subarc and the terminal subarc which intersect the boundary of the disk (shown in red).

Conceivably, the inside and outside of the triangle looks like the two train track diagrams pictured below in Figure 8.

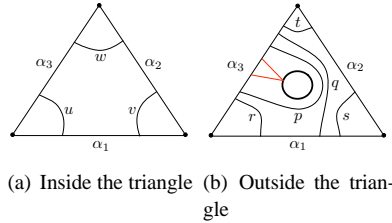


FIGURE 8.

However, we see that one of u, v, w must be zero, for if all of these are positive, then all of r, s, t must be zero (else, we would have a loop). This would imply that one of the sides of the triangle has no intersection points contradicting u, v, w all positive. Hence, one of u, v, w must be zero or equivalently, two of a_1, b_1, c_1 sum to give the third. Without any loss of generality, let's say $a_1 + b_1 = c_1$. With this assumption, we can conclude that the inside of the triangle is one of the two options shown in Figure 9.

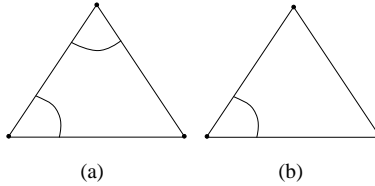


FIGURE 9.

It can be easily checked that the configuration in Figure 9(b) is the special case that $a_1 = 1, b_1 = 0, c_1 = 1$, and hence β acts as the identity on y_1 up to multiplication by Δ^{2k} . Assuming that a_1, b_1, c_1 are all positive as in Figure 9(a), we have the following six choices for the outside of the triangle shown in Figure 10.

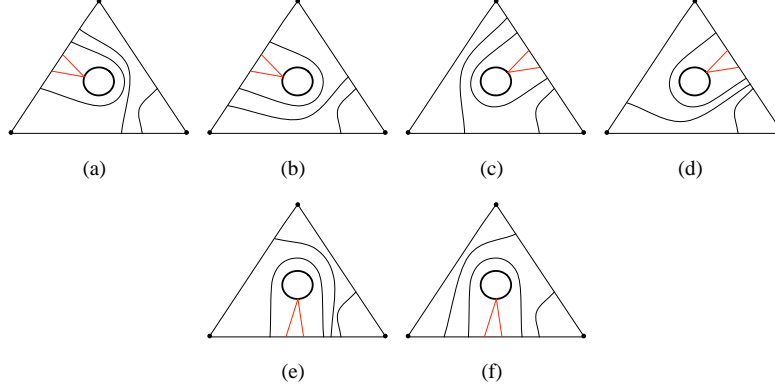


FIGURE 10.

Immediately, we can rule out the configurations in Figures 10(c) and 10(f) because each would imply that $c_1 = 0$. Now consider the configuration in Figure 10(d). If n_1, n_2, n_3 are the number of points on each side shown below in Figure 11, then $n_1 = a_1, n_2 = c_1$, and $n_1 + n_2 + n_3 = b_1$. Given that $a_1 + b_1 = c_1$, this gives that n_1 and n_3 are both equal to zero. This is a contradiction since a_1 is not equal to zero. We can also eliminate configuration 10(e) in a similar fashion (they are mirror images of each other).

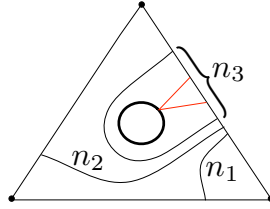


FIGURE 11.

Therefore, we are left with only two possibilities: 10(a) and 10(b). If m_1, m_2, m_3 are as shown below in Figure 12, then $m_1 = b_1, m_1 + m_2 = a_1$, and $m_2 + m_3 = c_1$. In particular, this configuration corresponds to the case that $a_1 > b_1$. It is easily checked that 10(b) corresponds to the case that $b_1 > a_1$, and if $a_1 = b_1$, then $m_2 = 0$. Hence, the only ambiguity is the amount of twisting that occurs in a neighborhood of ∂D_n . Thus, we can uniquely determine each $\beta(y_i)$ up to multiplication by powers of Δ^2 .

3.3. Veering information depends on k . Given $\beta \in B_3$, for each $i = 1, 2, 3$ we can construct y'_i in $\pi_1(D_n, p_0)$ so that the following hold:

- i. y'_i and $\beta(y_i)$ have the same geometric intersection numbers a_i, b_i, c_i ;
- ii. y'_i is to the right of y_i ; and

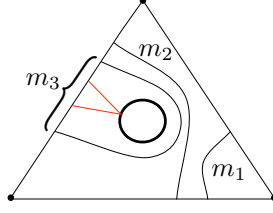


FIGURE 12.

iii. y'_i has minimal boundary twisting in the sense that $\Delta^{-2}(y'_i)$ is to the left of y_i . Furthermore, Theorem 1.2 guarantees that $\beta(y_i) = \Delta^{2k}(y'_i)$. Therefore, the sign of k tells us whether $\beta(y_i)$ is to the right of y_i . We summarize this in the following corollary.

Corollary 3.2. *Let $y'_i \in \pi_1(D_n, p_0)$ be as described above. Then $\beta(y_i) = \Delta^{2k}(y'_i)$. Moreover, $\beta(y_i)$ is to the right of y_i if and only if $k \geq 0$.*

Proof. This is immediate from the construction of y'_i and Theorem 1.2. \square

Now, how do we find the integer k ? Admittedly, this is the most cumbersome part of the process. After y'_i is found, we use the method described in Section 1.4 to calculate each corresponding column vector C_i of a matrix which we call $M(\beta')$. Here is where we use the faithfulness of the representation for $n = 3$. If we are lucky, this matrix will be $M(\beta)$, and hence $k = 0$. If we are not, we use our computing software of choice to multiply

$M(\beta')$ by powers of $M(\Delta^2) = \begin{pmatrix} t^2 & t^2 - t^3 & t^2 - t^3 \\ t - t^2 & t - t^2 + t^3 & t - t^2 \\ 1 - t & 1 - t & 1 - t + t^3 \end{pmatrix}$ until we reach $M(\beta)$.

The power of $M(\Delta^2)$ that gives $M(\beta)$ is equal to k .

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